

$SU(N)$ Skyrmions from Instantons*

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Abstract

Atiyah and Manton [1] have outlined a scheme to obtain approximations to the $SU(2)$ skyrmions from instantons in \mathbb{R}^4 . In this paper we apply this scheme to construct, in an explicit form, approximations to static spherically symmetric $SU(N)$ skyrmions with various baryon numbers. In particular we show how to obtain the skyrmions from instantons using harmonic maps into complex projective spaces.

1 Introduction

In this paper we construct a class of cylindrically symmetric $SU(N)$ instantons and calculate some of their properties, like the topological charges. This construction involves harmonic maps of the plane into \mathbb{CP}^{N-1} . The symmetry of the solutions determines the dependence of the fields on the three dimensional polar angles and leaves unknown only the dependence on the three dimensional radius (r) and the Euclidean time (τ). These solutions describe instantons with the same spatial location but centered at different times and with different scales.

Atiyah and Manton [1] have observed that computing the holonomy of $SU(2)$ instantons in \mathbb{R}^4 generates configurations in \mathbb{R}^3 which are good approximations to solutions of the Skyrme model. Here, we extend their construction to $SU(N)$ and derive explicitly some $SU(3)$ spherically symmetric skyrmions from the $SU(3)$ cylindrically symmetric instantons. In addition, we connect these skyrmion solutions with the ones obtained using the harmonic maps of S^2 to \mathbb{CP}^{N-1} [2].

Perhaps we should make it clear that there are several approaches to studying cylindrically symmetric instantons and spherically symmetric skyrmions. The main aim of this paper is not the construction of new instanton or skyrmion solutions, but rather to gain a better understanding of the correspondence between them and harmonic maps. In

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particular, using the Atiyah-Manton procedure [1], we obtain explicit closed form approximations for the profile functions of the $SU(N)$ Skyrme fields, which until now could only be determined numerically.

2 $SU(N)$ Instantons

Instantons are solutions of the $SU(N)$ Yang-Mills equations in \mathbb{R}^4 which are derived from the action functional

$$S = -\frac{1}{16\pi^2} \int \text{tr}(F_{ij}^2) d^4x \quad (2.1)$$

which is expressed in terms of topological charge units, ie $S \geq |k|$. Here k is the topological charge (or Pontryagin index) which counts the number of instantons of the configuration. [Since we are in the self-dual sector, ie $F = *F$, the topological charge k is given by (2.1)]. A_i , for $i = 0, 1, 2, 3$, is the $su(N)$ -valued gauge potential, with field strength $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ and covariant derivative $D_i = \partial_i + [A_i, \cdot]$. The finiteness of the action implies that the field strength must go to zero at spatial infinity, which means that the gauge field A_i must be a pure gauge at spatial infinity.

Variation of the action (2.1) gives the second order Yang-Mills equations

$$D_i F_{ij} = 0. \quad (2.2)$$

The starting point for our investigation is the introduction of the coordinates u, \bar{u}, z, \bar{z} on \mathbb{R}^4 . In terms of the usual spherical coordinates r, θ, φ the Riemann sphere variable is $z = e^{i\varphi} \tan(\theta/2)$, while $u = r + i\tau$. Using these coordinates the Yang-Mills equations (2.2) take the form

$$D_u((u + \bar{u})^2 F_{u\bar{u}}) + (1 + |z|^2)^2 (D_z F_{u\bar{z}} + D_{\bar{z}} F_{uz}) = 0 \quad (2.3)$$

$$D_{\bar{u}}((u + \bar{u})^2 F_{u\bar{u}}) - (1 + |z|^2)^2 (D_z F_{\bar{u}\bar{z}} + D_{\bar{z}} F_{\bar{u}z}) = 0 \quad (2.4)$$

$$D_u F_{\bar{u}z} + D_{\bar{u}} F_{uz} - \frac{1}{(u + \bar{u})^2} D_z((1 + |z|^2)^2 F_{z\bar{z}}) = 0 \quad (2.5)$$

$$D_u F_{\bar{u}\bar{z}} + D_{\bar{u}} F_{u\bar{z}} + \frac{1}{(u + \bar{u})^2} D_{\bar{z}}((1 + |z|^2)^2 F_{z\bar{z}}) = 0 \quad (2.6)$$

while the corresponding action (2.1) becomes

$$S = -\frac{1}{8\pi^2} \int \text{tr} \left(-4F_{u\bar{u}}^2 + \frac{8(1 + |z|^2)^2}{(u + \bar{u})^2} (|F_{uz}|^2 + |F_{u\bar{z}}|^2) - \frac{4(1 + |z|^2)^4}{(u + \bar{u})^4} F_{z\bar{z}}^2 \right) r^2 dr d\tau \frac{2i dz d\bar{z}}{(1 + |z|^2)^2}. \quad (2.7)$$

A complex gauge can, at least for the self-dual case, be chosen so that

$$A_{\bar{u}} = 0, \quad A_{\bar{z}} = 0. \quad (2.8)$$

The full equations (2.4) and (2.6) become then the total $\partial_{\bar{u}}$ - and $\partial_{\bar{z}}$ - derivative, respectively, of the self-dual Yang-Mills equation

$$\partial_{\bar{u}} A_u + \frac{(1 + |z|^2)^2}{(u + \bar{u})^2} \partial_{\bar{z}} A_z = 0 \quad (2.9)$$

while the other two become simply $F_{uz} = 0$, ie they define the gauge

$$A_z = H^{-1}H_z, \quad A_u = H^{-1}H_u \quad (2.10)$$

where $H \in SL(N, \mathbb{C})$ is a Hermitian matrix and subscripts denote partial differentiation.

To proceed further we need to briefly recall some results about harmonic maps of the two-dimensional \mathbb{CP}^{N-1} sigma model. See Zakrzewski [3] for a more detailed account of two-dimensional sigma models and their solutions.

2.1 Harmonic Maps

The harmonic map (or sigma model) equations for the \mathbb{CP}^{N-1} model are given by

$$[P_{z\bar{z}}, P] = 0 \quad (2.1)$$

where P is an $N \times N$ Hermitian projector.

One set of solutions to these equations are the instantons given by

$$P(f) = \frac{ff^\dagger}{|f|^2} \quad (2.2)$$

where $f(z)$ is an N -component column vector which is a holomorphic function of z and whose degree is equal to the topological charge of the sigma model. Another set of solutions are the anti-instantons, which have the same form but this time f is an anti-holomorphic function, and then the sigma model topological charge is minus the degree of f .

For $N = 2$ these are all the finite action solutions, but for $N > 2$ there are other non-instanton solutions. These can be described by introducing the operator Δ defined by its action on any vector $f \in \mathbb{C}^N$ as

$$\Delta f = \partial_z f - \frac{f(f^\dagger \partial_z f)}{|f|^2} \quad (2.3)$$

and then define further vectors $\Delta^k f$ by induction: $\Delta^k f = \Delta(\Delta^{k-1} f)$.

To proceed further we note the following useful properties of $\Delta^k f$ when f is holomorphic:

$$(\Delta^k f)^\dagger \Delta^l f = 0, \quad k \neq l \quad (2.4)$$

$$\partial_{\bar{z}} (\Delta^k f) = -\Delta^{k-1} f \frac{|\Delta^k f|^2}{|\Delta^{k-1} f|^2}, \quad \partial_z \left(\frac{\Delta^{k-1} f}{|\Delta^{k-1} f|^2} \right) = \frac{\Delta^k f}{|\Delta^{k-1} f|^2}. \quad (2.5)$$

These properties either follow directly from the definition of Δ or are easy to prove [3]. It is also convenient to define projectors P_k corresponding to the family of vectors $\Delta^k f$ as

$$P_k = P(\Delta^k f), \quad k = 0, \dots, N-1. \quad (2.6)$$

Applying Δ a total of $N - 1$ times to a holomorphic vector gives an anti-holomorphic vector, so that a further application of Δ gives the zero vector and hence no corresponding projector.

The projectors P_k are solutions of the harmonic map equations (2.1) and all solutions can be found in this way by starting with an appropriate holomorphic vector f . In the \mathbb{CP}^1 case the operator Δ converts a holomorphic vector to an anti-holomorphic vector, that is, instantons to anti-instantons and these are all the solutions in this case.

Note that the projectors obtained from this sequence always satisfy the relation $\sum_{k=0}^{N-1} P_k = 1$.

2.2 Constructing the Instantons

The self-dual Yang-Mills equation (2.9) after the gauge choice (2.10) is equivalent to the single equation for H

$$\partial_{\bar{u}}(H^{-1}H_u) + \frac{(1 + |z|^2)^2}{(u + \bar{u})^2} \partial_{\bar{z}}(H^{-1}H_z) = 0. \quad (2.1)$$

This is similar to an equation introduced by Jarvis [4] for studying monopoles, under a dimensional reduction of the time τ . As we are going to illustrate, exact $SU(N)$ instanton solutions of (2.1) can be obtained using harmonic maps, ie by assuming that the field H is of the form

$$H = \exp \left\{ g_0 \left(P_0 - \frac{1}{N} \right) + g_1 \left(P_1 - \frac{1}{N} \right) + \dots + g_{N-2} \left(P_{N-2} - \frac{1}{N} \right) \right\} \quad (2.2)$$

where $g_i = g_i(u, \bar{u})$ for $i = 0, \dots, N - 2$, are arbitrary functions of u and \bar{u} . Recall that the projector P_{N-1} is a linear combination of the other projectors plus the identity matrix, which is why it is not included in the above formula. The above ansatz is motivated by our recent study [5] of Bogomolny monopoles and their construction in terms of harmonic maps.

In order for our ansatz (2.2) to give solutions to (2.1), the harmonic maps used must have spherical symmetry — essentially the factors of $(1 + |z|^2)^2$ which appear in (2.1) must be cancelled. The required harmonic maps are obtained by applying the above procedure to the initial holomorphic vector

$$f = (f_{N-1}, \dots, f_j, \dots, f_0)^t, \quad \text{where} \quad f_j = z^j \sqrt{\binom{N-1}{j}} \quad (2.3)$$

and $\binom{N-1}{j}$ denote the binomial coefficients. For a discussion of the spherical symmetry of these maps see Ref. [5]. Here we merely point out that it is at least plausible that the required factors do indeed cancel since $|f|^2 = (1 + |z|^2)^{N-1}$. We shall illustrate this explicitly in the following with some examples.

SU(2) Case

There are simplifying special cases for which we are able to perform the construction explicitly, the easiest example being the rational map $f = (z, 1)^t$. For $N = 2$, there is only one profile function g_0 and our ansatz (2.2) reduces the self-dual equation (2.1) to the following differential equation

$$-(u + \bar{u})^2 g_{0u\bar{u}} + 2(e^{g_0} - 1) = 0. \quad (2.4)$$

Moreover, the topological charge k is given by

$$k = -\frac{1}{4\pi} \int \left(4g_{0u\bar{u}}^2 + \frac{4}{r^2} e^{g_0} |g_{0u}|^2 + \frac{1}{r^4} (e^{g_0} - 1)^2 \right) r^2 dr d\tau \quad (2.5)$$

where we have used the fact that $i \int dz d\bar{z} (1 + |z|^2)^{-2} = 2\pi$.

To actually solve (2.4), we first let $g_0 = 2 \ln(\frac{u+\bar{u}}{2}) + 2\rho_0$ (where ρ_0 is a new unknown function). Then equation (2.4) becomes

$$4\rho_{0u\bar{u}} = e^{2\rho_0} \quad (2.6)$$

which is the so-called Liouville equation and can be solved explicitly using conformal invariance. In this case, the function g_0 is

$$g_0 = 2 \ln \left(\frac{(u + \bar{u}) |dh/du|}{1 - |h|^2} \right). \quad (2.7)$$

Here h is, an analytic function of u , of the form (see Ref. [6])

$$h = \prod_{i=1}^{k+1} \frac{a_i - u}{\bar{a}_i + u} \quad (2.8)$$

and the a_i are an arbitrary set of complex numbers (some of them perhaps equal) constrained to have $\text{Re } a_i > 0$. Then (2.7) provide the most general solution of (2.1) with cylindrical symmetry and finite action. For general $k+1$, the total multiplicity of the zeros of h in the right half plane is always k ; therefore this solution describes k instantons. The imaginary part of the zeroes of h determines the location of the instantons along the time axis, while the real part determines the instanton scales [6].

For the special case where $h = (a_1 - u)^2 / (\bar{a}_1 + u)^2$ (for $a_1 = \lambda + i\lambda$) the topological charge k of our solution (2.7) is equal to one, ie the configuration consists of one instanton solution located at the origin with scale λ .

SU(3) Case

For $N = 3$ there are two profile functions g_0 and g_1 and equation (2.1) reduces to

$$\begin{aligned} -(u + \bar{u})^2 g_{0u\bar{u}} + 2(e^{g_1} - 1) + 2(e^{g_0 - g_1} - 1) &= 0 \\ -(u + \bar{u})^2 g_{1u\bar{u}} + 4(e^{g_1} - 1) - 2(e^{g_0 - g_1} - 1) &= 0. \end{aligned} \quad (2.9)$$

It is immediately clear that there is a symmetry under the interchange of indices $0 \leftrightarrow 1$, when applied simultaneously to g_i for $i = 0, 1$. As we will show later, this symmetry can be used to derive special instantons which involve a smaller number of profile functions and projectors.

In addition, the topological charge becomes

$$k = -\frac{1}{3\pi} \int \left\{ 4 \left(g_{0u\bar{u}}^2 + g_{1u\bar{u}}^2 - g_{0u\bar{u}} g_{1u\bar{u}} \right) + \frac{6}{r^2} e^{g_0 - g_1} \left(|g_{0u}|^2 + |g_{1u}|^2 - g_{0u} g_{1\bar{u}} - g_{0\bar{u}} g_{1u} \right) + \right. \\ \left. \frac{6}{r^2} e^{g_1} |g_{1u}|^2 + \frac{1}{r^4} \left(3e^{2(g_0 - g_1)} - 3e^{(g_0 - g_1)} + 3 + 3e^{2g_1} + 3e^{g_1} - 3e^{g_0} \right) \right\} r^2 dr d\tau \quad (2.10)$$

The profile functions equations that we obtain, ie (2.9), are related to those derived from the ansatz based approach of Bais et al [7] and the methods employed there can be adapted to solve for the functions explicitly. For the case in which the functions g_k are time independent, we have shown that [5], the equations (2.11) are precisely the equations for spherically symmetric monopoles.

To solve (2.9), we first let

$$g_0 = 2 \ln \left(\frac{(u + \bar{u})^2}{4} \right) + 2\rho_0 + 2\rho_1 \\ g_1 = 2 \ln \left(\frac{(u + \bar{u})}{2} \right) + 4\rho_0 - 2\rho_1 \quad (2.11)$$

where ρ_0 and ρ_1 are arbitrary functions of u and \bar{u} . Then (2.9) simplifies to

$$4\rho_{0u\bar{u}} = e^{4\rho_0 - 2\rho_1}, \quad 4\rho_{1u\bar{u}} = e^{4\rho_1 - 2\rho_0} \quad (2.12)$$

which are a set of coupled Liouville equations.

We have two sets of solutions to (2.12). The first one is just the maximal embedding of $SU(2)$ in $SU(3)$ and occurs when $\rho_0 = \rho_1$, that is for $g_0 = 2g_1$ (ie, using the symmetry). Then the system (2.12) simplifies to the Liouville equation and the solution is just Witten's solution [6] embedded in $SU(3)$, ie g_1 is given by (2.7). In this case, the topological charge (2.10) is exactly four times the $SU(2)$ one (given by (2.5)). Therefore, for the special case $h = (a_1 - u)^2 / (\bar{a}_1 + u)^2$, the configuration consists of four instantons — placed on top of each other in space.

The second set of solutions to (2.11) describes an irreducible $SU(3)$ instanton for which

$$\rho_0 = \frac{1}{2} \ln \left[\frac{3|dh/du|^2}{|h|^{1/3}(1 - |h|)^2(1 + 2|h|)} \right] \\ \rho_1 = \frac{1}{2} \ln \left[\frac{3|dh/du|^2}{|h|^{2/3}(1 - |h|)^2(2 + |h|)} \right] \quad (2.13)$$

where h is given by (2.8).

For the special case where $h = (a_1 - u)^2/(\bar{a}_1 + u)^2$ (for $a_1 = \lambda + i\lambda$) the topological charge (2.10) of our solution (2.11) is equal to two.

3 $SU(N)$ Skyrmions

The Skyrme model is a nonlinear field theory which provides a good description of low energy hadron physics. To have finite-energy configurations, one must require that the field $U(\vec{x}, t) \in SU(N)$ goes to a constant matrix (say 1) at spatial infinity: $U \rightarrow 1$ as $|\vec{x}| \rightarrow \infty$. This effectively compactifies the three-dimensional Euclidean space onto S^3 and hence implies that the Skyrme field can be considered as a map from S^3 into $SU(N)$; and therefore it can be classified by the third homotopy group $\pi_3(SU(N)) = Z$ or, equivalently, by the integer valued winding number

$$B = \frac{1}{24\pi^2} \int_{R^3} \varepsilon_{ijk} \text{tr} \left(\partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1} \right) d^3 \vec{x} \quad (3.1)$$

which is topological invariant. This winding number classifies the solitonic sectors in the model, and as Skyrme has argued [8], it may be identified with the baryon number B of the field configuration.

In the static limit, the energy of the Skyrme model is

$$E = \frac{1}{12\pi^2} \int_{R^3} \left\{ -\frac{1}{2} \text{tr} \left(\partial_i U U^{-1} \right)^2 - \frac{1}{16} \text{tr} \left[\partial_i U U^{-1}, \partial_j U U^{-1} \right]^2 \right\} d^3 \vec{x} \quad (3.2)$$

which is expressed in the same units as the baryon number. There is a lower bound on the energy of a given configuration in terms of the baryon number, ie $E \geq |B|$.

Since the model is not integrable explicit skyrmion solutions are not known and therefore, must be obtained by solving the equations numerically. In what follows using the Atiyah-Manton approach and harmonic maps we construct explicitly approximations to the $SU(N)$ skyrmions.

3.1 Harmonic Maps

Recently in [2], $SU(N)$ spherically symmetric skyrmion fields have been constructed from the harmonic maps of S^2 to \mathbb{CP}^{N-1} (which are *not* embeddings of the $SU(2)$ fields). In fact, the Skyrme field involves the introduction of $N - 1$ projectors, ie

$$U = \exp \left\{ i g_{0_{S_k}} \left(P_0 - \frac{1}{N} \right) + i g_{1_{S_k}} \left(P_1 - \frac{1}{N} \right) + \dots + i g_{(N-2)_{S_k}} \left(P_{N-2} - \frac{1}{N} \right) \right\} \quad (3.3)$$

where $g_{i_{S_k}} = g_{i_{S_k}}(r)$ for $i = 0, \dots, N - 2$ are the profile functions.

Moreover, the energy (3.2) becomes

$$E = \frac{1}{6\pi} \int r^2 dr \left\{ -\frac{1}{N} \left(\sum_{i=0}^{N-2} \dot{g}_{i_{S_k}} \right)^2 + \sum_{i=0}^{N-2} \dot{g}_{i_{S_k}}^2 + \frac{1}{2r^2} \sum_{k=1}^{N-1} \left(\dot{g}_{k_{S_k}} - \dot{g}_{(k-1)_{S_k}} \right)^2 D_k + \frac{2}{r^2} \sum_{k=1}^{N-1} D_k \right\}$$

$$+\frac{1}{4r^4}\left(D_1^2+\sum_{k=1}^{N-2}(D_k-D_{k+1})^2+D_{N-1}^2\right)\} \quad (3.4)$$

where $D_k = k(N-k)(1 - \cos(g_{k_{Sk}} - g_{(k-1)_{Sk}}))$.

In addition, the topological charge (3.1) takes the form

$$B = \frac{1}{2\pi} \sum_{i=0}^{N-2} (i+1)(N-i-1) \left(g_{i_{Sk}} - g_{(i+1)_{Sk}} - \sin(g_{i_{Sk}} - g_{(i+1)_{Sk}}) \right) \Big|_{r=0}^{r=\infty}. \quad (3.5)$$

As $g_{i_{Sk}}(\infty) = 0$ (required for the finiteness of the energy) the only contributions to the topological charge comes from $g_i(0)$.

This way a family of exact spherically symmetric solutions of the $SU(N)$ Skyrme model has been obtained. In fact for each $SU(N)$ model the Skyrme field involving $N-1$ projectors leads to an exact solutions involving $N-1$ profile functions $g_{i_{Sk}}$. These profile functions $g_{i_{Sk}}$, which satisfy $N-1$ coupled nonlinear ordinary differential equations and can be solved numerically, are exhibited in Ref. [2]. Next we will derive explicitly analytic forms for $g_{i_{Sk}}$ in (3.3) which are good approximations to the ones obtained numerically in [2].

3.2 Derivation of Skyrmions

It has been proposed by Atiyah and Manton [1] that a finite-dimensional manifold of Skyrme fields can be generated from $SU(2)$ self-dual Yang-Mills fields. The main idea of the Atiyah-Manton scheme is to construct the holonomy

$$U(\vec{x}) = \mathcal{T} \exp \left(- \int_{-\infty}^{\infty} A_{\tau}(\vec{x}, \tau) d\tau \right) \quad (3.1)$$

where \mathcal{T} denotes time-ordering and $x = (\vec{x}, \tau)$ denotes the time line through \vec{x} .

The fundamental topological result is that if we consider this holonomy as a Skyrme field then it has baryon number $B = k$, where k is the topological charge of the gauge potential A_i (ie, instanton number). The fields can be explicitly computed in some special cases, by integrating simple expressions.

The gauge field A_{τ} , using the harmonic map ansatz (2.2) and due to the gauge choice (2.8), takes the form

$$A_{\tau} \equiv iA_u = -ig_{0u}(\frac{1}{N} - P_0) - ig_{1u}(\frac{1}{N} - P_1) - \dots - ig_{(N-2)u}(\frac{1}{N} - P_{N-2}). \quad (3.2)$$

By substituting (3.2) into the Atiyah-Manton formula (3.1) and comparing with (3.3) we see that the profile functions of the instanton and Skyrme fields are related. In fact, the profile functions of the $SU(N)$ Skyrme fields $g_{i_{Sk}}$ can be determined analytically through the relation

$$g_{i_{Sk}} = - \int_{-\infty}^{\infty} g_{iu}(r, \tau) d\tau \quad (3.3)$$

for $i = 0, \dots, N - 2$. [Recall that, these profile functions (3.3) are approximations to the ones obtained numerically in [2]].

Next, we will determine analytically the $g_{i_{Sk}}$ profile functions of the Skyrme field, for the simplest cases of $SU(2)$ and $SU(3)$. This way, we construct a class of spherically symmetric skyrmions and calculate some of their properties, such as their energies and baryon numbers. Recall that, their baryon number is equal to the topological charge of the corresponding instanton configuration they are derived from.

SU(2) Case

There is only one profile function in this case, $g_{0_{Sk}}$, which can be obtained from (3.3) for g_0 given by (2.7). This gives the well-known one spherically symmetric skyrmion solution, with

$$g_{0_{Sk}} = 2\pi \left[1 - (1 + \lambda^2/r^2)^{-1/2} \right]. \quad (3.4)$$

This Skyrme field has first been obtained in [1]. It can be shown that the configuration consists of one skyrmion (due to (3.5)) and the energy obtained from (3.4) has minimum at $\lambda^2 = 2.11$ and is equal to $E = 1.2432$, ie within 1% of the numerically determined value 1.232.

SU(3) Case

For $N = 3$ there are two profile functions $g_{0_{Sk}}$ and $g_{1_{Sk}}$ which again can be evaluated explicitly using (3.3). [Recall that we have two set of solutions for the instanton functions].

When $g_0 = 2g_1$ for g_1 given by (2.7), we end up having one profile function for the Skyrme field which coincides with the profile function of a single $SU(2)$ skyrmion, ie

$$g_{1_{Sk}} = 2\pi \left[1 - (1 + \lambda^2/r^2)^{-1/2} \right]. \quad (3.5)$$

Here we note that as $g_{0_{Sk}}(0) = 4\pi$ and $g_{1_{Sk}}(0) = 2\pi$ the topological charge (3.5) of our solution is four (so is the instanton number). The energy obtained from (3.4) of this configuration is exactly four times the energy of the one $SU(2)$ skyrmion, ie $E = 4 \times 1.2432$. This compares with the energy of the numerically solution 4×1.232 determined in [2]. [Again the minimum occurs at $\lambda^2 = 2.11$].

In the case where the two instanton profile functions are given by equations (2.11) and (2.13), after performing the integration (3.3) we get

$$\begin{aligned} g_{0_{Sk}} &= -2\pi \left(1 - \frac{\lambda + 3r}{2\sqrt{9\lambda^2 + 6\lambda r + 9r^2}} + \frac{\lambda - 3r}{2\sqrt{9\lambda^2 - 6\lambda r + 9r^2}} \right) \\ g_{1_{Sk}} &= -\pi \left(1 + \frac{\lambda + 3r}{\sqrt{9\lambda^2 + 6\lambda r + 9r^2}} + \frac{2(\lambda - 3r)}{\sqrt{9\lambda^2 - 6\lambda r + 9r^2}} \right). \end{aligned} \quad (3.6)$$

Since $g_{0_{Sk}}(0) = g_{1_{Sk}}(0) = -2\pi$ the baryon number (3.1) is two (recall, $k = 2$); the interpretation of this solution is therefore that it contains two skyrmions. The approximate energy

obtained from (3.4) is then $E = 2.39815$ which is in good agreement with the true value 2.3764 determined in [2]. This solutions has, first, been obtained by Balachandran et al [9].

4 Conclusion

We have studied in some detail the construction of $SU(N)$ instanton from harmonic maps. Explicit solutions have been obtained in the case of cylindrical symmetry and we have shown how these solutions involve harmonic maps of the plane into \mathbb{CP}^{N-1} . In addition, we have generate approximate solutions of the $SU(N)$ Skyrme model using the Atiyah-Manton procedure, ie by computing instanton holonomies. The corresponding skyrmions are spherically symmetric, their energies are in good agreement with their exact value and their baryon number is equal to the instanton number.

The same approach can be applied to generate other soliton approximations in lower dimensions (see for example, Ref. [10]), leading to the fact that they might be many connections between solitons and instantons in varying spacetime dimensions.

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